

Nonlinear mirror instability

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Slow dynamical changes in magnetic-field strength and invariance of the particles' magnetic moments generate ubiquitous pressure anisotropies in weakly collisional, magnetized astrophysical plasmas. This renders them unstable to fast, small-scale mirror and firehose instabilities, which are capable of exerting feedback on the macroscale dynamics of the system. By way of a new asymptotic theory of the early nonlinear evolution of the mirror instability in a plasma subject to slow shearing or compression, we show that the instability does not saturate quasilinearly at a steady, low-amplitude level. Instead, the trapping of particles in small-scale mirrors leads to nonlinear secular growth of magnetic perturbations, $\delta B/B \propto t^{2/3}$. Our theory explains recent collisionless simulation results, provides a prediction of the mirror evolution in weakly collisional plasmas and establishes a foundation for a theory of nonlinear mirror dynamics with trapping, valid up to $\delta B/B = O(1)$.

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Introduction. Dynamical, weakly collisional high- β plasmas develop pressure anisotropies with respect to the magnetic field as a result of the combination of slow changes in magnetic-field strength B and conservation of the first adiabatic invariant of particles $\mu = v_\perp^2/2B$. This renders them unstable to fast (ion cyclotron timescale Ω_i^{-1}), small-scale (ion gyroscale ρ_i) firehose, mirror and ion cyclotron instabilities [1–6], whose observational signatures have been reported in the solar wind [7, 8] and planetary magnetosheaths [9–15]. These instabilities are also thought to be excited in energetic astrophysical environments such as the intracluster medium (ICM) [16–20], the vicinity of accreting black holes [21–23], or the warm ionized interstellar medium [24, 25], producing strong dynamical feedback at macroscales, with critical astrophysical implications [26–31]. A self-consistent description of the multiscale physics of such plasmas requires understanding how these instabilities saturate nonlinearly.

Let us consider a typical situation in which slow changes in B due to shearing, compression or expansion of the plasma at large (“fluid”) scales build up ion pressure anisotropy $\Delta_i \equiv (p_i^\perp - p_i^\parallel)/p_i^\perp$, driving the plasma through either the ion firehose instability boundary ($\Delta_i < -2/\beta_i$, with $\beta_i = 8\pi p_i/B^2$) in regions of decreasing field, or the mirror instability boundary ($\Delta_i \gtrsim 1/\beta_i$) in regions of increasing field. This triggers exponential growth on timescales up to Ω_i^{-1} , much faster than the shearing timescale S^{-1} . The separation between these timescales implies that the instabilities always operate close to threshold and regulate the levels of pressure anisotropy in the plasma nonlinearly. However, how they achieve that in the face of the slowly changing B constantly generating more pressure anisotropy, remains an open question.

In the simplest case of the parallel firehose instability, the growth of magnetic perturbations $\delta \mathbf{B}$ leads to an increase of the average (r.m.s.) field strength and perpendicular pressure, which drives the anisotropy back to marginality, $\Delta_i(t) \rightarrow$

$-2/\beta_i$. If a weakly unstable initial state $\Delta_{i0} - 2/\beta_i < 0$ is postulated with no further driving of Δ_i , quasilinear theory [32] predicts saturation at a steady, low amplitude $\delta B/B \sim |\Delta_{i0} + 2/\beta_i|^{1/2} \ll 1$. However, the shearing or expansion process that drove the plasma through the instability boundary in the first place, must ultimately become important again once quasilinear relaxation has pushed the system sufficiently close back to marginality. When such continued driving is accounted for, asymptotic theory [33, 34] predicts secular growth of perturbations as $\delta B/B \propto t^{1/2}$ up to $\delta B/B = O(1)$ (cf. [35]), a very different outcome from steady-state, low-amplitude saturation.

The nonlinear dynamics of the mirror instability [36–39] in a weakly collisional shearing (or compressing) plasma driven through its instability boundary is much more involved and has only recently been explored numerically [40, 41]. In this Letter, we show that weakly nonlinear mirror modes in such conditions do not saturate quasilinearly at a steady, low amplitude either, but continue to grow secularly as $\delta B/B \propto t^{2/3}$. To do this, we introduce a new asymptotic theory in the spirit of earlier theoretical work [33, 34, 42], in which the combined effects of weak collisionality, large-scale shearing, quasilinear relaxation [32, 43, 44], particle trapping [45–48] and finite ion Larmor radius (FLR) are all self-consistently retained.

Asymptotic theory. We consider the simplest case of a plasma consisting of cold electrons [54] and hot ions of mass m_i , charge $q_i = Ze$, and thermal velocity $v_{thi} = \sqrt{2T_i/m_i}$, coupled to the electromagnetic fields \mathbf{E} and \mathbf{B} . The dynamics is governed by the non-relativistic Vlasov-Maxwell system,

$$\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \nabla f_i + \frac{q_i}{m_i} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_i}{\partial \mathbf{v}} = C[f_i], \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad \mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}, \quad (2)$$

and Ohm's law describing the force balance for electrons,

$$\mathbf{E} + \frac{\mathbf{u}_i \times \mathbf{B}}{c} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi Z n_i}. \quad (3)$$

Here, $f_s(t, \mathbf{r}, \mathbf{v})$, $n_s(t, \mathbf{r}) = \int f_s d^3\mathbf{v}$ and $\mathbf{u}_s(t, \mathbf{r}) = \int \mathbf{v} f_s d^3\mathbf{v}$ are, respectively, the distribution function, number density and mean velocity of species $s = (i, e)$, $\mathbf{j} = e n_e (\mathbf{u}_i - \mathbf{u}_e)$ is the total current density given the quasineutrality condition $n_e = Z n_i$. In the following, we use the ion peculiar velocity $\mathbf{v}' = \mathbf{v} - \mathbf{u}_i$ as the velocity-space variable and will henceforth drop the primes. Taking the first moment of Eq. (1) and using Eqs. (2)-(3), we obtain the ion momentum equation

$$\frac{d\mathbf{u}_i}{dt} = -\frac{\nabla \cdot \mathbf{P}_i}{m_i n_i} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi m_i n_i}, \quad (4)$$

where $d/dt = \partial/\partial t + \mathbf{u}_i \cdot \nabla$ and $\mathbf{P}_i = m_i \int \mathbf{v} \mathbf{v} f_i d^3\mathbf{v}$ is the ion pressure tensor. Introducing $\hat{\mathbf{b}} \equiv \mathbf{B}/B$, using Eqs. (2)-(3), we obtain the evolution equation for the field strength:

$$\frac{d \ln B}{dt} = \hat{\mathbf{b}} \hat{\mathbf{b}} : \nabla \mathbf{u}_i - \nabla \cdot \mathbf{u}_i - \frac{\hat{\mathbf{b}}}{B} \cdot \nabla \times \left(\frac{\mathbf{j} \times \mathbf{B}}{Z n_i} \right). \quad (5)$$

Our derivation is based on an asymptotic expansion of these equations. The separation between the slow magnetic-field-stretching timescale and the fast instability timescale implies that the distance to instability threshold $\Gamma \sim \Delta_i - 1/\beta_i$ must remain small, which provides us with a natural expansion parameter. In order to study the dynamics in this regime, we start from an already weakly unstable situation and order $\Gamma = O(\varepsilon^2)$, with $\varepsilon \ll 1$. We then construct a ‘‘maximal’’ ordering (summarized in Eqs. (8)-(12)) retaining ion FLR, collisional, quasilinear and trapping effects, as well as the effect of continued slow shearing. Following [39], we order the time and spatial scales of mirror modes as $\gamma \sim \varepsilon^2 k_{\parallel} v_{\text{th}i}$, $k_{\perp} \sim \varepsilon^{-1} k_{\parallel}$, $\rho_i^{-1} \sim \varepsilon^{-2} k_{\parallel}$, where γ is the instability growth rate, $(\mathbf{k}_{\perp}, k_{\parallel})$ the typical perturbation wavenumbers (defined with respect to the unperturbed field), $\rho_i = v_{\text{th}i\perp}/\Omega_i$, and $\Omega_i^{-1} = (m_i c)/q_i B \sim \varepsilon^{-2} k_{\parallel} v_{\text{th}i}$. The ion distribution function is expanded as $f_i = f_{0i} + f_{2i} + \delta f$, where f_{0i} provides the required pressure anisotropy to pin the system at the threshold, f_{2i} provides an extra $O(\varepsilon^2)$ anisotropy to drive the system away from it, and δf contains mirror perturbations. We also expand $\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}$ and $\mathbf{u}_i = \mathbf{u}_{0i} + \delta \mathbf{u}_i$, where \mathbf{B}_0 and \mathbf{u}_{0i} have no instability-scale variations, \mathbf{u}_{0i} is the slow, large-scale shearing/compressive motion, and $\delta \mathbf{B}$ and $\delta \mathbf{u}_i$ are the mirror perturbations.

The ordering of δf , $\delta \mathbf{B}$, $\delta \mathbf{u}_i$ and of the remaining timescales is guided by physical considerations. The critical pitch-angle $\xi = v_{\parallel}/v$ below which particles get trapped by magnetic fluctuations is $\xi_{\text{tr}} = (\delta B/B_0)^{1/2}$ and the corresponding bounce frequency is $\omega_B \sim k_{\parallel} v_{\text{th}i} \xi_{\text{tr}}$. To retain their contribution in our calculation, we order $\omega_B \sim \gamma \sim \varepsilon^2 k_{\parallel} v_{\text{th}i}$, which provides us with the ordering $\delta B/B_0 = O(\varepsilon^4)$ (plus higher-order terms). For consistency of the ε expansion, we must order $\delta f = O(\varepsilon^4)$, $\delta \mathbf{u}_i = O(\varepsilon^5)$ and higher. Averaging Eq. (5) over instability

scales and ignoring quadratic nonlinearities, we obtain

$$\frac{d \ln B_0}{dt} = \hat{\mathbf{b}}_0 \hat{\mathbf{b}}_0 : \nabla \mathbf{u}_{0i} - \nabla \cdot \mathbf{u}_{0i} \equiv S. \quad (6)$$

Subtracting Eq. (6) from Eq. (5), we find that

$$\frac{d}{dt} \frac{\delta B}{B_0} = \hat{\mathbf{b}}_0 \hat{\mathbf{b}}_0 : \nabla \delta \mathbf{u}_i \quad (7)$$

to all relevant orders [55]. We order $S \equiv d/dt(\ln B_0)$ the same size as $d/dt(\delta B/B_0) \sim \varepsilon^6 k_{\parallel} v_{\text{th}i}$ so as to be able to investigate how a slow change in field strength affects the dynamics. The aforementioned timescale separation S/Ω_i is now related to ε through $\varepsilon \sim (S/\Omega_i)^{1/8}$ ($\varepsilon \sim 0.01$ for the ICM [34]).

A separate asymptotic treatment is required for low-pitch-angle resonant particles, which develop a velocity-space boundary layer and evolve into a separate population of trapped particles on the instability timescale (the process is reminiscent of ‘‘nonlinear Landau damping’’ [47, 49, 50]). As can be seen by considering a simple Lorentz pitch-angle scattering operator [51], $C[f_i] = (v_{ii}/2) \partial_{\xi} [(1 - \xi^2) \partial_{\xi} f_i]$, this results in a boost of their effective collisionality, $v_{ii, \text{eff}} \sim v_{ii}/\xi_{\text{tr}}^2 \gg v_{ii}$ for $\xi < \xi_{\text{tr}} \ll 1$. To retain this effect, we order $v_{ii, \text{eff}} \sim \gamma \sim \omega_B$, or $v_{ii} \sim k_{\parallel} v_{\text{th}i} (\delta B/B_0)^{3/2} \sim \varepsilon^6 k_{\parallel} v_{\text{th}i}$ (this preserves the low-collisionality assumption in the sense that the rest of the distribution relaxes on a timescale $1/v_{ii} \gg 1/\gamma$). The maximal mirror ordering is summarized as follows:

$$f_i = f_{0i} + f_{2i} + \delta f_{4i} + \dots, \quad (8)$$

$$\gamma \sim \varepsilon^2 k_{\parallel} v_{\text{th}i}, \quad \Omega_i \sim \varepsilon^{-2} k_{\parallel} v_{\text{th}i}, \quad S \sim v_{ii} \sim \varepsilon^6 k_{\parallel} v_{\text{th}i}, \quad (9)$$

$$\rho_i^{-1} \sim \varepsilon^{-2} k_{\parallel}, \quad k_{\perp} \sim \varepsilon^{-1} k_{\parallel}, \quad (10)$$

$$\mathbf{B} = \mathbf{B}_0 + \delta B_4^{\parallel} \hat{\mathbf{b}}_0 + \delta \mathbf{B}_5^{\perp} + \dots, \quad \mathbf{u}_i = \mathbf{u}_{0i} + \delta \mathbf{u}_{5i}^{\perp} + \dots, \quad (11)$$

$$\xi_{\text{tr}} \sim (\delta B_4^{\parallel}/B_0)^{1/2} \sim \varepsilon^2, \quad \omega_B \sim v_{ii, \text{eff}} \sim \gamma \sim \varepsilon^2 k_{\parallel} v_{\text{th}i}. \quad (12)$$

Taking the three lowest non-trivial orders of Eq. (1), we first find that f_{0i} , f_{2i} , δf_{4i} are gyrotropic. Expanding and gyroaveraging Eq. (1) up to $O(\varepsilon^4)$ then gives δf_{4i} in terms of the mirror perturbation $\delta B_4^{\parallel}/B_0$, from which the perturbed scalar pressures δp_{4i}^{\perp} and $\delta p_{4i}^{\parallel}$ are derived (note that resonant/trapped particles are not involved at this stage). Taking the perpendicular projection of Eq. (4) at the lowest order $O(\varepsilon^3)$, we obtain the threshold condition for the mirror instability [39]:

$$\Gamma_0 = -\frac{2 m_i}{p_{0i}^{\perp}} \int \frac{v_{\perp}^4}{4} \frac{\partial f_{0i}}{\partial v_{\perp}^2} \Big|_{v_{\perp}} d^3\mathbf{v} - \frac{2}{\beta_{0i}^{\perp}} - 2 = 0. \quad (13)$$

Next, we expand and gyroaverage Eq. (1) to three further orders, up to $O(\varepsilon^7)$. This tedious calculation yields FLR corrections and resonant effects (not shown, see [42] for an almost identical procedure) and provides us with explicit expressions for δf_{5i} and δf_{6i} , from which we obtain higher-order elements of \mathbf{P}_i , $\delta \mathbf{p}_{5i}^{\perp \parallel} \equiv m_i \int \mathbf{v}_{\perp} v_{\parallel} \delta f_{5i} d^3\mathbf{v}$ and $\delta \mathbf{P}_{6i}^{\perp \perp}$, in terms of δB_4^{\parallel} and δB_6^{\parallel} . No new information arises from Eq. (4) at $O(\varepsilon^4)$. Using these results and Eq. (13) in the perpendicular projection of

Eq. (4) at $O(\varepsilon^5)$, we derive the pressure balance condition:

$$\left[\Gamma_2 + \frac{3}{2} \rho_*^2 \nabla_\perp^2 - \left(\frac{p_{0i}^\perp - p_{0i}^\parallel}{p_{0i}^\perp} + \frac{2}{\beta_{0i}^\perp} \right) \frac{\nabla_\parallel^2}{\nabla_\perp^2} \right] \nabla_\perp \frac{\delta \tilde{B}_4^\parallel}{B_0} = \nabla_\perp \frac{\delta \tilde{p}_{6i}^{\perp(\text{res})}}{p_{0i}^\perp}, \quad (14)$$

where the resonant/trapped particle pressure is

$$\delta \tilde{p}_{6i}^{\perp(\text{res})} = m_i \int_{|\xi| < \xi_{\text{tr}}} \frac{v_\perp^2}{2} \delta \tilde{f}_{4i}^{(\text{res})} d^3 \mathbf{v}, \quad (15)$$

$\delta f_{4i}^{(\text{res})}$ is the resonant part of the perturbed distribution function, the second-order distance to instability threshold is

$$\Gamma_2 = - \frac{2 m_i}{p_{0i}^\perp} \left(\int \frac{v_\perp^4}{4} \frac{\partial f_{2i}}{\partial v_\parallel^2} \Big|_{v_\perp} d^3 \mathbf{v} + \int_{|\xi| < \xi_{\text{tr}}} \frac{v_\perp^4}{4} \frac{\partial \overline{\delta f_{4i}^{(\text{res})}}}{\partial v_\parallel^2} \Big|_{v_\perp} d^3 \mathbf{v} \right) - \frac{2 p_{2i}^\perp}{p_{0i}^\perp}, \quad (16)$$

and the effective Larmor radius is

$$\rho_*^2 = \frac{\rho_i^2}{12} \frac{m_i}{p_{0i}^\perp v_{\text{th}i\perp}^2} \int \left(-v_\perp^6 \frac{\partial f_{0i}}{\partial v_\parallel^2} \Big|_{v_\perp} - 3 v_\perp^4 f_{0i} \right) d^3 \mathbf{v}. \quad (17)$$

Tildes denote fluctuating (zero field-line average) parts of the perturbed fields and overlines denote line averages. The l.h.s. of Eq. (14) describes the non-resonant response [56]. Γ_2 and $\delta \tilde{p}_{6i}^{\perp(\text{res})}$ depend on the regime considered. However, both only involve $\delta f_{4i}^{(\text{res})}$ because restricting the integration to ξ_{tr} brings in an extra $O(\varepsilon^2)$ smallness, whereas FLR corrections only start to affect the distribution function at $O(\varepsilon^6)$ within our expansion. Thus, $\delta \tilde{f}_{4i}^{(\text{res})}$ and $\delta \tilde{p}_{6i}^{\perp(\text{res})}$ can be directly calculated from the much simpler drift-kinetic equation which, in (μ, v_\parallel) variables, reads [52]:

$$\frac{df_i}{dt} + v_\parallel \nabla_\parallel f_i = -\mu B (\nabla \cdot \mathbf{b}) \frac{\partial f_i}{\partial v_\parallel} + C[f_i] \quad (18)$$

to all orders relevant to our calculation (here $E_\parallel = 0$ because the electrons are cold). For resonant particles, $v_\parallel \sim \varepsilon^2 v_{\text{th}i}$ and $\partial \delta f_{4i}^{(\text{res})} / \partial v_\parallel \sim (\varepsilon^{-2} / v_{\text{th}i}) \delta f_{4i}^{(\text{res})}$, so the expansion of Eq. (18) at the first non-trivial order $O(\varepsilon^6)$ is

$$\frac{d \delta f_{4i}^{(\text{res})}}{dt} + v_\parallel \nabla_\parallel \delta f_{4i}^{(\text{res})} = \mu B_0 \left(\frac{\nabla_\parallel \delta B_4^\parallel}{B_0} \right) \left(\frac{\partial f_{0i}}{\partial v_\parallel} + \frac{\partial \delta f_{4i}^{(\text{res})}}{\partial v_\parallel} \right) + C[\delta f_{4i}^{(\text{res})}]. \quad (19)$$

We have omitted the collision term $C[f_{0i}]$: it gives a $O(\varepsilon^4)$ correction to the line-averaged pressure on the instability timescale that can be absorbed into Γ_0 .

Linear and quasilinear regimes. Neglecting the nonlinear and collision terms in Eq. (19) and taking its space-time Fourier transform, we obtain the linear solution:

$$\delta \tilde{f}_{4i\mathbf{k}}^{(\text{res})} = \frac{\mu i k_\parallel \delta \tilde{B}_{4\mathbf{k}}^\parallel}{\gamma_L + i k_\parallel v_\parallel} \frac{\partial f_{0i}}{\partial v_\parallel}, \quad (20)$$

where γ_L is the linear instability growth rate. Using Eq. (15) to compute $\delta \tilde{p}_{6i}^{\perp(\text{res})}$ and substituting the result into Eq. (14), the

classical linear mirror dispersion relation [39] is recovered:

$$\gamma_L = \sqrt{\frac{2}{\pi}} |k_\parallel| v_* \left[\Gamma_2 - \frac{3}{2} \rho_*^2 k_\perp^2 - \left(\frac{p_{0i}^\perp - p_{0i}^\parallel}{p_{0i}^\perp} + \frac{2}{\beta_{0i}^\perp} \right) \frac{k_\parallel^2}{k_\perp^2} \right], \quad (21)$$

with the effective thermal speed

$$v_*^{-1} = -\sqrt{2\pi} \frac{2m_i}{p_{0i}^\perp} \int \frac{v_\perp^4}{4} \frac{\partial f_{0i}}{\partial v_\parallel^2} \Big|_{v_\perp} \delta(v_\parallel) d^3 \mathbf{v}. \quad (22)$$

Because of the resonance, $\delta \tilde{f}_{4i}^{(\text{res})}$ develops a velocity-space boundary layer on the timescale $\sim 1/\gamma_L$, resulting in a correction to the line-averaged distribution function that satisfies:

$$\frac{d \overline{\delta f_{4i}^{(\text{res})}}}{dt} = -\mu B_0 \left(\frac{\nabla_\parallel \delta \tilde{B}_4^\parallel}{B_0} \right) \frac{\partial \overline{\delta f_{4i}^{(\text{res})}}}{\partial v_\parallel} + C[\overline{\delta f_{4i}^{(\text{res})}}]. \quad (23)$$

Assuming a monochromatic perturbation for simplicity and using Eq. (20) to calculate the line-averaged nonlinear term on the r.h.s. of Eq. (23), we recover the resonant quasilinear diffusion equation [32]:

$$\frac{d \overline{\delta f_{4i}^{(\text{res})}}}{dt} = \frac{\partial}{\partial v_\parallel} \left[\frac{2 (\mu B_0)^2 k_\parallel^2 \gamma_L}{\gamma_L^2 + (k_\parallel v_\parallel)^2} \left(\frac{\delta \tilde{B}_4^\parallel}{B_0} \right)^2 \frac{\partial f_{0i}}{\partial v_\parallel} \right] + C[\overline{\delta f_{4i}^{(\text{res})}}]. \quad (24)$$

The effect of the first term on the r.h.s. of Eq. (24) is to relax Γ_2 (see Eq. (16)) by flattening the total averaged distribution function at low ξ , thereby decreasing the growth rate [43, 44].

Trapping regime. Quasilinear relaxation ceases to be the dominant saturation mechanism once particle trapping becomes dynamically significant ($\omega_B \sim k_\parallel v_{\text{th}i} (\delta B/B)^{1/2} \sim \gamma_L$). Indeed, due to the growth of $\delta B/B$ and quasilinear reduction of the growth rate $\partial/\partial t \ll \gamma_L$ for $t \gg 1/\gamma_L$, (i) the system eventually reaches a bounce-dominated regime, $\omega_B \gg \partial/\partial t$, and (ii) collisional and shearing effects, however slow their timescales are, compared to the initial linear instability timescale, inevitably become important after a few instability times (hence the maximal ordering Eqs. (8)-(12)). To elicit these effects, we rewrite Eq. (18) in $(\mu, E = v^2/2)$ variables:

$$\frac{df_i}{dt} \pm \sqrt{2(E - \mu B)} \frac{\partial f_i}{\partial \ell} = -\mu \frac{dB}{dt} \frac{\partial f_i}{\partial E} + C[f_i], \quad (25)$$

where ℓ is the distance along the perturbed field line. Expanding Eq. (5) and Eq. (25) at $O(\varepsilon^6)$, we obtain

$$\begin{aligned} \frac{d \delta f_{4i}^{(\text{res})}}{dt} \pm \sqrt{2(E - \mu B)} \frac{\partial \delta f_{4i}^{(\text{res})}}{\partial \ell} &= -\mu B_0 \frac{d}{dt} \frac{\delta B_4^\parallel}{B_0} \frac{\partial f_{0i}}{\partial E} \\ &\quad - \mu B_0 \left(\frac{d \ln B_0}{dt} + \frac{d}{dt} \frac{\delta B_4^\parallel}{B_0} \right) \frac{\partial \delta f_{4i}^{(\text{res})}}{\partial E} + C[\delta f_{4i}]. \end{aligned} \quad (26)$$

Here $C[f_{0i}]$ and $-\mu B_0 (d \ln B_0 / dt) (\partial f_{0i} / \partial E)$ have been discarded for the same reason as in Eq. (19). Note that both $\partial \delta f_{4i}^{(\text{res})} / \partial E$ and $\partial \delta f_{4i}^{(\text{res})} / \partial \mu$ are $O(1)$ because of the velocity-space boundary layer in $|\xi| < \xi_{\text{tr}} = O(\varepsilon^2)$.

We anticipate that magnetic fluctuations will grow secularly as $\delta B_4^\parallel \sim \delta B_4^\parallel(t_L)(t/t_L)^s$, with $s > 0$, for $t \gg t_L \sim 1/\gamma_L (\sim 1/\omega_B)$, and introduce a secondary ordering parameter $\chi = (t_L/t)^{s/2} \ll 1$, so now $\delta B_4^\parallel/B_0 = O(\varepsilon^4/\chi^2)$ and $\xi_{tr} \sim (\delta B_4^\parallel/B_0)^{1/2} = O(\varepsilon^2/\chi) \gg \varepsilon^2$. The instantaneous nonlinear growth rate is $\gamma_{NL} \sim \partial/\partial t \sim 1/t = O(\varepsilon^2\chi^{2/s})$, so the new ordering guarantees $\omega_B \sim k_\parallel v_{thi} \xi_{tr} \gg \gamma_{NL}$. For trapped particles to play a role in the nonlinear evolution, their pressure in Eq. (14) must be taken to be of the same order as the instability-driving term, $\Gamma_2(\delta B_4^\parallel/B_0) \sim \delta p_{0i}^\perp/p_{0i}^\perp = O(\varepsilon^2/\chi^2)$. Given that $\delta p_{0i}^\perp(\text{res}) \sim \xi_{tr} \delta f_{4i}^{(\text{res})}$, we must therefore order $\delta f_{4i}^{(\text{res})} = O(\varepsilon^4/\chi)$. Expanding Eq. (26) to lowest order $O(\varepsilon^6/\chi^2)$, we find that the distribution function of the trapped particles is homogenized along the field lines within the traps: $\partial \delta f_{4i}^{(\text{res})}/\partial \ell = 0$. Therefore, $\delta f_{4i}^{(\text{res})} = \langle \delta f_{4i}^{(\text{res})} \rangle + \delta f_{4i}^{(\text{res})'}$, where $\delta f_{4i}^{(\text{res})'} \ll \langle \delta f_{4i}^{(\text{res})} \rangle$ and $\langle \bullet \rangle = \oint \bullet d\ell$ denotes a bounce average between bounce points ℓ_1 and ℓ_2 defined by the relation $E = \mu B(\ell_1) = \mu B(\ell_2)$. Looking at the next orders of Eq. (26), we find that the first term on the r.h.s. (the betatron term linear in perturbations) is $O(\varepsilon^6\chi^{2/s-2})$, while the time derivative on the l.h.s. and the r.h.s. term quadratic in perturbations are $O(\varepsilon^6\chi^{2/s-1})$, so quasi-linear effects are subdominant. The terms involving $d \ln B_0/dt$ and collisions are $O(\varepsilon^6\chi)$. For Eq. (26) to have a solution at $O(\varepsilon^6\chi)$, we see that $s = 2/3$ is required, so $\delta B_4^\parallel/B_0 \propto t^{2/3}$. The resulting equation for $\delta f_{4i}^{(\text{res})'}$ is

$$\pm \frac{\partial \delta f_{4i}^{(\text{res})'}}{\partial \ell} = -\frac{\mu B_0}{\sqrt{2(E-\mu B)}} \left(\frac{d}{dt} \frac{\delta B_4^\parallel}{B_0} \frac{\partial f_{0i}}{\partial E} + \frac{d \ln B_0}{dt} \frac{\partial \langle \delta f_{4i}^{(\text{res})} \rangle}{\partial E} \right) + C[\langle \delta f_{4i}^{(\text{res})} \rangle]. \quad (27)$$

Using a Lorentz operator and bounce averaging, we obtain

$$\left\langle \frac{\mu B_0}{\sqrt{2(E-\mu B)}} \frac{d}{dt} \frac{\delta B_4^\parallel}{B_0} \right\rangle \frac{\partial f_{0i}}{\partial E} = -\frac{d \ln B_0}{dt} \left\langle \frac{\mu B_0}{\sqrt{2(E-\mu B)}} \right\rangle \frac{\partial \langle \delta f_{4i}^{(\text{res})} \rangle}{\partial E} + \frac{v_{ii}}{B_0} \frac{\partial}{\partial \mu} \left(\mu \left\langle \sqrt{2(E-\mu B)} \right\rangle \frac{\partial \langle \delta f_{4i}^{(\text{res})} \rangle}{\partial \mu} \right). \quad (28)$$

Physical behavior and temporal evolution. This equation has taken some effort to derive but is fairly transparent physically. It represents a competition between perpendicular betatron cooling of the equilibrium distribution due to the local decrease of the magnetic field in the deepening mirror traps (the l.h.s. of Eq. (28)), the perpendicular betatron heating of the trapped-particle population associated with the increasing mean field B_0 (the first term on the r.h.s.), and their collisional isotropization (the second term on the r.h.s.). In the weakly collisional, unsheared regime ($v_{ii} \neq 0$, $d \ln B_0/dt \equiv S = 0$), the balance is between betatron cooling and collisions. In the collisionless, shearing regime ($v_{ii} = 0$, $S > 0$), it is instead between betatron cooling (of the bulk distribution in mirror

perturbations) and heating (of the perturbed distribution in the growing mean field). In order for the system to stay marginal in the face of continued driving and/or collisional relaxation, magnetic perturbations have to continue growing. A simple physical interpretation of the collisionless case is that trapped particles regulate the evolution so as to “see” effectively a constant total magnetic field.

Solutions of Eqs. (14)-(15)-(28) can be found in the form $\delta B_4^\parallel/B_0 = \mathcal{A}(t)\mathcal{B}(\ell)$. Using Eq. (28), this implies $\langle \delta f_{4i}^{(\text{res})} \rangle = \alpha \mathcal{A} (d\mathcal{A}/dt) \mathcal{H}[(E - \mu B_0)/\mathcal{A}(t)] \partial f_{0i}/\partial E$, where $\alpha = 1/S$ if $v_{ii} = 0$ and $1/v_{ii}$ if $S = 0$ (\mathcal{H} also depends on the functional form of $\mathcal{B}(\ell)$). Then, from Eq. (15), $\delta p_{0i}^\perp(\text{res}) = \alpha \mathcal{A}^{3/2} (d\mathcal{A}/dt) \mathcal{F}(\ell)$. Equation (14) will have solutions if $\alpha \mathcal{A}^{1/2} (d\mathcal{A}/dt) = \Lambda \Gamma_2$, with Λ a constant of order unity. As anticipated in our discussion of the secondary ordering (in χ), this implies that perturbations grow secularly as

$$\mathcal{A}(t) = (\Lambda \Gamma_2 S t)^{2/3} \text{ and } \mathcal{A}(t) = (\Lambda \Gamma_2 v_{ii} t)^{2/3} \quad (29)$$

in the shearing-collisionless regime and unsheared-collisional regime, respectively. This result is formally valid for times $S t, v_{ii} t = O(\varepsilon^4)$ [57]. The $t^{2/3}$ time dependence also holds in mixed regimes ($v_{ii} \neq 0$, $S \neq 0$). Λ and $\mathcal{B}(\ell)$ (which is not sinusoidal) are obtained by solving a nonlinear eigenvalue equation involving trapping integrals. A detailed classification of the solutions of this equation lies beyond the scope of this Letter.

Conclusion. Equation (28) and the secular growth of the nonlinear mirror instability, $\delta B/B \propto t^{2/3}$, in both collisional and collisionless regimes, are the main results of this work. Thus, we appear to be approaching a theory in which trapping effects, higher amplitude nonlinearities [42, 47, 48, 53] and relaxation of anisotropy through anomalous particle scattering [40] blend together harmoniously. The results make manifest the importance of particle trapping [45, 46]. Numerical simulations [40] confirm $t^{2/3}$ secular growth of mirror perturbations in a collisionless, shearing plasma, with saturation amplitudes $\delta B/B = O(1)$ independent of S (cf. [41]).

The weakly collisional, weakly shearing regimes studied in this Letter occur in many natural environments [8, 21, 34] and are increasingly the focus of attention in the context of high-energy astrophysical plasmas [33, 34, 40, 41]. The emergence of finite-amplitude magnetic mirrors with scales smaller than the mean free path lends credence to the idea that microscale instabilities regulate processes such as heat conduction [26], viscosity, heating [27–29] and dynamo [30, 31] in such plasmas, and therefore profoundly alter their large-scale energetics and dynamics.

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[1] M. N. Rosenbluth, LANL Report No. LA-2030 (1956).

- [2] S. Chandrasekhar *et al.*, Proc. R. Soc. A **245**, 435 (1958).
- [3] E. N. Parker, Phys. Rev. **109**, 1874 (1958).
- [4] A. A. Vedenov and R. Z. Sagdeev, Sov. Phys. Dokl. **3**, 278 (1958).
- [5] L. I. Rudakov and R. Z. Sagdeev, Sov. Phys. Dokl. **6**, 415 (1961).
- [6] S. P. Gary, J. Geophys. Res. **97**, 8523 (1992).
- [7] P. Hellinger *et al.*, Geophys. Res. Lett. **33**, L09101 (2006).
- [8] S. D. Bale *et al.*, Phys. Rev. Lett. **103**, 211101 (2009).
- [9] R. L. Kaufmann *et al.*, J. Geophys. Res. **75**, 4666 (1970).
- [10] G. Erdös and A. Balogh, J. Geophys. Res. **101**, 1 (1996).
- [11] N. André *et al.*, Geophys. Res. Lett. **29**, 1980 (2002).
- [12] S. P. Joy *et al.*, J. Geophys. Res. **111**, A12212 (2006).
- [13] V. Génot *et al.*, Ann. Geophys. **27**, 601 (2009).
- [14] T. S. Horbury and E. A. Lucek, J. Geophys. Res. **114**, 5217 (2009).
- [15] J. Soucek and C. P. Escoubet, Ann. Geophys. **29**, 1049 (2011).
- [16] A. C. Fabian, Annu. Rev. Astron. Astr. **32**, 277 (1994).
- [17] C. L. Carilli and G. B. Taylor, Annu. Rev. Astron. Astr. **40**, 319 (2002).
- [18] F. Govoni and L. Feretti, Int. J. Mod. Phys. D **13**, 1549 (2004).
- [19] A. A. Schekochihin *et al.*, Astrophys. J. **629**, 139 (2005).
- [20] J. R. Peterson and A. C. Fabian, Phys. Rep. **427**, 1 (2006).
- [21] E. Quataert, in *Probing the Physics of Active Galactic Nuclei by Multiwavelength Monitoring*, edited by B. M. Peterson, R. S. Polidan, R. W. Pogge, ASP Conf. Series **224**, p. 71 (2001).
- [22] R. Narayan and E. Quataert, Science **307**, 77 (2005).
- [23] O. Blaes, Space Sci. Rev., Online First (2013) [arXiv:1304.4879].
- [24] K. M. Ferrière, Rev. Mod. Phys. **73**, 1031 (2001).
- [25] A. N. Hall, Mon. Not. R. Astron. Soc. **190**, 353 (1980).
- [26] B. D. G. Chandran & S. C. Cowley, Phys. Rev. Lett. **80**, 3077 (1998).
- [27] P. Sharma *et al.*, Astrophys. J. **637**, 952 (2006).
- [28] P. Sharma *et al.*, Astrophys. J. **667**, 714 (2007).
- [29] M. W. Kunz *et al.*, Mon. Not. R. Astron. Soc. **410**, 2446 (2011). A. A. Schekochihin and S. C. Cowley, Phys. Plasmas **13**, 056501 (2006).
- [30] A. A. Schekochihin and S. C. Cowley, Phys. Plasmas **13**, 056501 (2006).
- [31] F. Mogavero and A. A. Schekochihin, Mon. Not. R. Astron. Soc. **440**, 3226 (2014).
- [32] V. D. Shapiro and V. I. Shevchenko, Sov. Phys. JETP **18**, 1109 (1964).
- [33] A. A. Schekochihin *et al.*, Phys. Rev. Lett. **100**, 081301 (2008).
- [34] M. S. Rosin *et al.*, Mon. Not. R. Astron. Soc. **413**, 7 (2011).
- [35] L. Matteini, S. Landi, P. Hellinger, M. Velli, J. Geophys. Res., **111**, A10101 (2006).
- [36] M. Tajiri, J. Phys. Soc. Japan **22**, 1482 (1967).
- [37] A. Hasegawa, Phys. Fluids **12**, 2642 (1969).
- [38] D. J. Southwood and M. G. Kivelson, J. Geophys. Res. **98**, 9181 (1993).
- [39] P. Hellinger, Phys. Plasmas **14**, 082105 (2007).
- [40] M. W. Kunz *et al.*, Phys. Rev. Lett., **112**, 205003 (2014).
- [41] M. A. Riquelme *et al.*, submitted to Astrophys. J. (2014) [arXiv:1402.0014].
- [42] F. Califano *et al.*, J. Geophys. Res. **113**, A08219 (2008).
- [43] O. A. Pokhotelov *et al.*, J. Geophys. Res. **113**, A04225 (2008).
- [44] P. Hellinger *et al.*, Geophys. Res. Lett. **36**, L06103 (2009).
- [45] M. G. Kivelson and D. J. Southwood, J. Geophys. Res. **101**, 17365 (1996).
- [46] F. G. E. Pantellini, J. Geophys. Res. **103**, 4798 (1998).
- [47] Y. N. Istomin *et al.*, Phys. Plasmas **16**, 062905 (2009).
- [48] O. A. Pokhotelov *et al.*, Ann. Geophys. **28**, 2161 (2010).
- [49] J. Dawson, Phys. Fluids **4**, 869 (1961).
- [50] T. O’Neil, Phys. Fluids **8**, 2255 (1965).
- [51] P. Helander and D. J. Sigmar, Collisional Transport in Magnetized Plasmas, Cambridge: Cambridge University Press (2002).
- [52] R. M. Kulsrud, in *Handbook of Plasma Physics*, edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1983), Vol. 1, p. 115.
- [53] E. A. Kuznetsov *et al.*, Phys. Rev. Lett. **98**, 235003 (2007).
- [54] Formally, we first expand the electron Vlasov equation in $\sqrt{m_e/m_i}$ and then take the limit $T_e \ll T_i$, so electrons still stream along the field very fast. This is purely a matter of analytical convenience: our results also hold for hot, isothermal electrons.
- [55] A Hall term $-(\rho_i v_{thi\perp}/\beta_{0i}^\perp) \hat{\mathbf{b}}_0 \cdot [\nabla_\perp \times (\nabla_\parallel \delta \mathbf{B}_s^\perp/B_0)]$ is formally present in the induction equation at the maximum order $O(\varepsilon^6)$ needed here. However, it can be proven to be zero because of the particular polarization of mirror modes [42] and has therefore been summarily discarded in Eq. (7) to simplify the algebra.
- [56] Nonlinearities quadratic in $\delta B/B$ are negligible here because of the weakly nonlinear ordering $\delta B/B \sim \Gamma^2$, which differs from $\delta B/B \sim \Gamma$ used in [42].
- [57] These results do not apply to the frequently discussed “idealized” case $v_{ii} = S = 0$, because continued growth of fluctuations was assumed to derive Eq. (28). In that regime, fluctuations should instead settle in a steady state $\delta B/B \sim \Gamma^2$ through quasilinear relaxation, possibly after a few transient bounce oscillations [47]. However, only a small amount of collisions or continued shearing/compression is required for the present theory to apply: these effects are bound to become dynamically important after a few instability times, once quasilinear relaxation has reduced the instability drive Γ sufficiently.